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The order of buying and selling: Multiple equilibria in the housing market

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Abstract

Moving between owner-occupied houses requires both buying and selling. During the Great Recession, the majority of movers sold their old houses before buying new ones, while in the preceding years the majority first bought new houses. In an equilibrium search model, by choosing the order of buying and selling, households affect the composition of buyer and seller types in the market. Because of their different outside options, different types bargain for different prices. Since prices have an impact on the incentives to enter the market as buyer or seller, a complementarity in the order of buying and selling exists. The resulting multiple equilibria can explain observed differences in trading volumes. Moreover, when all movers first buy and therefore own two houses for some time, the fraction of people paying double housing expenses is *lower* than when households enter as buyers and sellers simultaneously, due to a smaller time to sale.

Keywords: Housing market, Search, Multiple equilibria, Trading volume, Time to sale

JEL Classification: R31

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1 Introduction

Houses are illiquid assets. Due to search frictions, buying and selling takes time. Moreover, moving between owner-occupied houses requires both buying *and* selling. The fact that moving requires two transactions results in two potential problems for a household. On the one hand, households that buy a new house but have problems selling their old house, suffer from double housing expenses. Households that sell their old house before finding a new one, on the other hand, are stuck in temporary housing in between. It depends on the tightness of the market to what extent being in either of these states is likely to last long.

Households realize these potential problems, and try to deal with them. One of the few options that is available to an individual household with a desire to move, is to choose the order of buying and selling. Households can first search to buy a new house, first search to sell their old house, or do both at the same time. However, if households collectively choose one of these search strategies, they affect the composition of buyer and seller types in the market. Different types of buyers and sellers have different outside options, and therefore bargain for different prices. This paper shows that first buying and then selling can generate the composition of buyers and sellers that makes this order of buying and selling individually rational to households. If, however, households would simultaneously enter the market as sellers and buyers, they would create the market conditions that would make this alternative search strategy rational. The same mechanisms apply in an equilibrium in which households first sell and then buy. The resulting multiple equilibria can explain empirically observed phases of high and low trading volumes in the housing market. In general, the equilibria can be ranked in terms of welfare, as different equilibria are characterized by different probabilities of moving out of inferior situations.

The data in table 1 can be interpreted as evidence for different search strategies before and during the Great Recession. The first column of the table provides the fractions of buyers in the Netherlands who, before buying a new house, lived in owner-occupied

housing. The second column presents the fraction that lived in rental housing.¹ Data are only available for five years, but interestingly, for periods before and during the Great Recession. The third column shows the total number of transactions in the Netherlands. The data show that the number of transactions before the Great Recession - in 1998, 1999, and 2000 - is significantly larger than the number of transactions during the Great Recession - in 2010 and 2011. Interestingly, these periods are associated with quite different fractions of buyers from owner-occupied and rental housing. In the period before the Great Recession, the majority of buyers previously lived in owner-occupied housing, whereas after the Great Recession, the majority of buyers lived in rental housing. These data can be interpreted in multiple ways.² The interpretation of this paper is that buyers choose different search strategies in periods of high and periods of low liquidity.

Year	Fraction of owners	Fraction of renters	Number of transactions
1998	0.53	0.47	192622
1999	0.58	0.42	204538
2000	0.61	0.39	189358
2010	0.47	0.53	126127
2011	0.47	0.53	120739

Table 1: Fractions of buyers previously owning and renting a house, and the total number of transactions in the Netherlands. Data from Statistics Netherlands, only available for five years.

Three other papers explicitly recognize that it matters whether a household first buys or first sells a house. The first paper, Anglin (2004), focuses on the implications of this issue for the length of time required by a household to move, and its empirical estimation (by the proportional hazard model). Although he presents some extensions that can make

¹These fractions only refer to the total of buyers that already participated on the housing market (existed as households) and of which their previous housing is known, so that the fractions sum up to one. These buyers constitute the majority of buyers, and are the population that actually have different search options.

²Unfortunately, the data (on fractions, not on transactions) on 1998, 1999, and 2000 are from a different data set than those on 2010 and 2011. The first data set considers households searching to buy a house, whereas the second set covers households that actually bought a house. Consequently, a simple difference in search effort between renters and owners can explain the different fractions. On the other hand, the correlation between the total number of searchers and the actual number of transactions, within those three years, is 0.96. Secondly, the different fractions could also result from a changing housing supply, either in composition or relative to the population. However, the housing supply relative to the number of households remained constant over these years, and the relative supply of rental housing did not increase.

an individual household move sooner, Anglin doesn't discuss the equilibrium aspects of the decision to first sell or to first buy a house. The second paper, by Maury and Tripier (2010), fills this gap to a large extent by showing that price dispersion is likely in equilibrium. It has a very similar setup as my paper, but they do not allow search strategies to affect market tightness. By fixing the probability to buy and to sell at an equal rate, they exclude the important feedback mechanism of the order of buying and selling. In general, in- and outflow rates will differ across equilibria. As a result, welfare can no longer be determined only by the value of a household satisfied with her house, and equilibrium rankings can be overturned. Moreover, fixing the same in- and outflow rates across equilibria boils down to assuming a different housing supply for each. From this perspective, Maury and Tripier's 'multiple equilibria' differ in fundamentals. Finally, Anenberg and Bayer (2013) show that the 'joint buyer-seller problem' amplifies booms and busts that result from shocks to the flow of new buyers. This 'joint buyer-seller problem' is modeled by sellers in one segment of the housing market that constitute the potential buyers in a second segment of the market. While very similar in motivation to my paper, Anenberg and Bayer limit complementarities by ruling out moving from the second to the first segment. Consequently, they do not study the existence of multiple equilibria, but provide a model rich enough to be estimated on empirical data. Only my paper explains phases of high and low liquidity in the housing market with a purely endogenous mechanism.

The seminal Wheaton (1990) and the more recent Piazzesi and Schneider (2009) also consider a housing market with random search, but do not allow households to have a choice about how and when to move. This paper extends their contributions to an equilibrium search model in which households can choose to first look for a new house, or simultaneously to search for a buyer for their old house. In contrast, in Wheaton (1990), households have to buy a new house before they can sell their old house, whereas in Piazzesi and Schneider (2009), households have to sell their old house before they can buy a new one.

In Krainer (2001), households can accumulate an infinite number of houses, essentially making the choice to search to sell and to search to buy independent decisions. As a

result, in an equilibrium with trade households always do both at the same time. In my model, households cannot own more than two houses. On top of that, households that own two houses pay an additional fee, capturing double housing expenses. While a first house can be financed by a mortgage without financial frictions, such frictions kick in when a household buys a second house. Indeed assuming that moral hazard problems increase with the amount borrowed, a first house can be financed at the common discount rate (or paid out of pocket, with opportunity costs discounted at the same rate), whereas a second house can only be financed at an additional flow cost. Moral hazard problems are prohibitively large for buying a third house. In this way I rule out pure down-payment effects, at the core of the argument in for instance Stein (1995), but still capture financial constraints in a parsimonious manner.

By focusing on the impact of different search strategies, I provide a mechanism that can endogenously create more or less desperate households in the housing market. For that reason, equilibrium price dispersion is a common phenomenon in this model, even though households are *ex ante* homogeneous. In this way, I endogenize the motivational shocks in Albrecht, Anderson, Smith, and Vroman (2007), which exogenously turn relaxed buyers and sellers into desperate ones. As a result, also in my model expected prices conditional on time to sale fall with time spent on the market. Moreover, my endogenous mechanism allows for a richer variety in the composition of buyer and seller types, which gives rise to a new source of equilibrium multiplicity. Besides, I keep track of the equilibrium stocks and flows of buyer and seller types, instead of introducing them exogenously at a constant and equal rate, to explain the observed differences in trading volume between the Great Recession and the years preceding it.

2 Setup of the housing market

In this section I present the setup of an equilibrium search model of the owner-occupied housing market. I follow Piazzesi and Schneider (2009) in assuming a competitive rental

market next to the owner-occupied housing market, while the latter is characterized by search frictions. The model describes short-run equilibria in the sense that the supply of housing is constant, and I only analyze steady states. Besides, I only consider symmetric pure-strategy equilibria.

2.1 Equilibrium overview

Time is continuous, the population consists of a unit mass of households, and $h \in (0, 2)$ stands for the measure of houses available for owner-occupied housing. I abstract from heterogeneity in houses, apart from idiosyncratic characteristics that give rise to search frictions. The unit mass of households consists of four types: households satisfied with the characteristics of the house they own (high valuation owners), households unsatisfied with the characteristics of the house they own (low valuation owners), households that own two houses (double owners), and households that own no house, but rent temporary housing (renters). Define the measure of renters in the population as μ_R , the measure of low valuation owners as μ_L , the measure of high valuation owners as μ_H , and the measure of double owners as μ_D . Houses are owned by high valuation, low valuation or double owners, where the latter own two houses, so that $h = \mu_H + \mu_L + 2\mu_D$. Renters live in apartments outside housing supply h , not in the μ_D unoccupied houses.³ Finally, define the measure of buyers as μ_B and the measure of sellers as μ_S .

Individual households make transitions between the types, as shown in figure 1. At some positive exogenous Poisson rate $\eta > 0$ a high valuation owner receives a preference shock, turning her into a low valuation owner. The preference shock captures exogenous reasons for a desire to move, such as finding a new job elsewhere. A low valuation owner is no longer satisfied with her house, but still owns it. She would like to move to a new house that she is satisfied with, but moving requires finding a new house, and finding a buyer for her old house. The figure shows that there are two ways out of the low valuation state,

³Often mortgage contracts do not allow owners to rent out their houses. Double owners may also not *want* to rent out their old houses, as this would hinder selling them.

via double ownership or via renting. The low valuation owner becomes a double owner if she first buys a new house, whereas she becomes a renter if she first sells her old house. All sellers find interested buyers at the same (endogenous) Poisson rate p , and all buyers find houses to their liking at the same (endogenous) Poisson rate q . As discussed, due to financial limitations, a double owner cannot buy more than two houses. For simplicity, I assume that a double owner is no longer subject to preference shocks, so that she has a high valuation of exactly one of her houses.⁴ Her only option is thus to sell the house she doesn't like anymore, and to become a high valuation owner. Similarly, the only option available to a renter is to buy a new house. This route also results in high valuation ownership, and thus makes the household subject to preference shocks again.

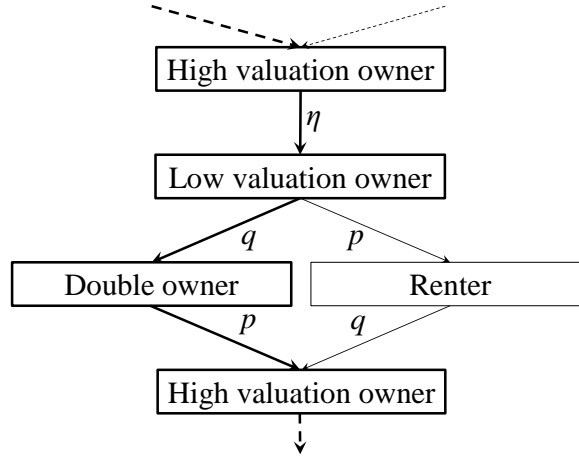


Figure 1: Transitions and their respective Poisson rates of the simultaneous equilibrium

Because I only consider symmetric pure-strategy equilibria, if one household of any type searches, then all households of that type search. If low valuation owners enter the market both as sellers and as buyers, and if double owners and renters search as well, then we have an equilibrium with two types of buyers and two types of sellers. Both renters and low valuation owners search for a new house, so that the total measure of buyers in this equilibrium is $\mu_B = \mu_R + \mu_L$. Both double owners and low valuation owners search to

⁴As a rationale for this assumption, assume there are two categories of houses - e.g. houses in the city and houses in the countryside - and that shocks make households' preferences only switch between these two categories. Houses within each category still differ in their idiosyncratic characteristics, so that not all houses of one category are a match, but the preference shocks do not affect the idiosyncratic tastes, so that an owner of two houses is always satisfied with one of them even if she is subject to preference shocks.

sell their old house, so that the measure of sellers is $\mu_S = \mu_D + \mu_L$. Since low valuation households search as buyers and sellers simultaneously, I refer to this equilibrium as the simultaneous equilibrium.

If low valuation owners only enter the market as buyers, then all of them move to high valuation ownership via double ownership. In this case only the left route of figure 1 is used. This means that in steady state, renters disappear. For that reason, the steady state measure of buyers is equal to the measure of low valuation owners, and the steady state measure of sellers is the measure of double owners. This ‘first-buy-then-sell’ equilibrium is similar to the sequential search model of Wheaton (1990).⁵

On the other hand, if low valuation owners only enter the market as sellers, then all of them move to the high valuation state via rental apartments. As a result, double ownership disappears, and only the right route of 1 is used. Steady state sellers are low valuation households, while steady state buyers are renters. This ‘first-sell-then-buy’ equilibrium is similar to the sequential search model of Piazzesi and Schneider (2009). However, it follows from Lemma 1 that the two sequential search strategy equilibria can never coexist in steady state for the same housing supply.

Lemma 1. *Housing supply $h \in (0, 2)$ determines which of the sequential search strategy steady states can exist. For $0 < h < 1$ the ‘first-buy-then-sell’ equilibrium cannot exist, while for $1 < h < 2$ the ‘first-sell-then-buy’ equilibrium cannot exist. For $h = 1$ neither of the sequential steady states can exist.*

Proof. It follows from the distribution of houses and the sum of the four fractions of households being one that

$$h - 1 = \mu_D - \mu_R. \tag{1}$$

Consequently, for $0 < h < 1$ the left-hand side is negative, so that renters cannot disappear,

⁵For a rental market to continue to exist in this equilibrium, one can assume two types of renters. Analogous to the distinction between unemployed and non-participants in the labor market, the participating renters described above are those that search to buy a house, while a second category of renters has no ambition to enter the housing market and never searches. This last type of renter can be ignored in an analysis of the owner-occupied housing market.

while double owners can. For $1 < h < 2$ the left-hand side is positive, so that double owners cannot disappear, while renters can. Since the market collapses if they disappear both, for $h = 1$ no sequential steady states exist. \square

Upon switching between states, the buyer pays the transaction price to the seller. For simplicity, I follow Piazzesi and Schneider (2009) in assuming that sellers have all bargaining power, making take-it-or-leave-it offers under perfect information about the buyer's type. In case of multiple buyer types, take-it-or-leave-it offers from sellers therefore generally create endogenous price dispersion. This price dispersion thus results from the different outside options of different types of buyers, not from the idiosyncratic housing characteristics. For that reason, the price P_i is indexed by the buyer type $i \in \{L, R\}$ involved in the transaction. To avoid the Diamond paradox, if buyers have no bargaining power they cannot have search costs.⁶ Without search costs, low valuation owners always enter the market as buyers, so that the assumption of take-it-or-leave-it offers from sellers is not suitable to study the 'first-sell-then-buy' equilibrium. However, given Lemma 1 it is harmless to study only one sequential equilibrium at a time.

To allow for a 'first-buy-then-sell' equilibrium given Lemma 1, discussing this equilibrium I combine the assumption of full bargaining power for sellers with a housing supply $h \in (1, 2)$. Although this implies that the long side of the market has all bargaining power, note that the higher transaction probabilities of the short side are incorporated in its outside option. I show that a 'first-buy-then-sell' equilibrium can exist for the same fundamentals as a simultaneous search strategy equilibrium. The same mechanism applies for the coexistence of the 'first-sell-then-buy' and the simultaneous equilibrium (with search costs for buyers instead of sellers, take-it-or-leave-it offers from buyers, and $h \in (0, 1)$), but in this paper I omit the presentation of the 'first-sell-then-buy' equilibrium. From now on I shall therefore refer to the 'first-buy-then-sell' equilibrium as the sequential equilibrium. First I present the value functions of the different household types.

⁶Otherwise, buyers would incur costs that are sunk at the moment of trade, without being able to claim some share of the surplus to cover their costs. Buyers would retreat from search, and the market would collapse.

2.2 Values for four types of households

Households are risk neutral, discount the future at rate r , and care about consumption and housing services (also denoted in consumption units). All types of households enjoy basic housing services u . High valuation owners receive additional flow benefits v on top of u . However, at rate $\eta > 0$ they turn into low valuation owners. In principle a high valuation owner could also sell her house or buy a second house, but I will show that this doesn't happen in equilibrium. The flow value of being a high valuation owner is then

$$rV_H = v + u - \eta(V_H - V_L). \quad (2)$$

A low valuation owner wants to find a new house and to find a buyer for her old house. Because the housing market is characterized by search frictions, moving takes time. Search is random, and comes at a flow cost c for sellers only. Define market tightness as $\theta \equiv \mu_B/\mu_S$. The number of matches M follows from a (homogenous of degree one) Cobb-Douglas matching function, taking the measure of buyers and sellers as inputs: $M = m(\mu_B, \mu_S) = m_0 \mu_B^\alpha \mu_S^{1-\alpha}$. The number of matches per seller, or the rate of finding a potential buyer for one's house, is thus given by $M/\mu_S = m(\theta, 1) = m_0(\mu_B/\mu_S)^\alpha$. Below I derive conditions such that all matches between potential buyers and sellers result in transactions, so that in equilibrium the transaction rates p and q are equal to the matching rates for sellers and buyers respectively. Since the number of sales must always equal the number of purchases, we have that $q = p/\theta$. In contrast to Piazzesi and Schneider (2009) and Maury and Tripier (2010), I do not assume that buying and selling takes the same amount of time, so that in general $p \neq q$. I only analyze equilibria with positive transaction rates, since multiplicity of equilibria from market collapse is already well-known.⁷

Without search costs to buyers, a low valuation owner always enters the market as buyer. The question is whether she also enters as seller, first transacting on the side of the market on which the opportunity arises the soonest. The latter search strategy is pursued if and

⁷The classic reference is Diamond (1982).

only if the costs c do not exceed the benefits, which depend on the expected price for a house. Since the buyer type is not known in advance, the expected price is denoted by P_i . The flow value of a low valuation owner is then

$$rV_L = u + q(V_D - V_L - P_L) + \max[0, p(V_R - V_L + P_i) - c]. \quad (3)$$

This equation clearly shows that there are two ways out of the low valuation state, via double ownership or via renting. Rental housing matched to the preferences of households is immediately available, so that renters enjoy housing services $v + u$. However, because the rental market is competitive and suppliers of rental housing can freely enter and exit the market, housing services enjoyed by the household are paid in rent $R = v + u$. A renting household can search for a new home though, so that her flow value is

$$rV_R = q(V_H - V_R - P_R). \quad (4)$$

As discussed, owning a second house comes at an additional flow cost F capturing the double housing expenses that a household owning two houses incurs. On top of that, a double owner enjoys the basic housing services u for only one house, since she can live in only one at a time (and cannot rent out the other). On the other hand, a double owner always enjoys v . Because a double owner cannot buy a third house, her only option is to sell the house she doesn't like anymore. The flow value of a double owner is therefore

$$rV_D = v + u - F + \max[0, p(V_H - V_D + P_i) - c]. \quad (5)$$

The order of buying and selling is important because the household wants to avoid being stuck in temporary housing or paying double housing expenses. Because any flow benefits from housing services cancel out against a competitive rental rate, temporary housing is clearly undesirable. To model paying double housing expenses as a sufficiently serious problem, I make Assumption 1 throughout the paper.

Assumption 1. *The double housing expenses flow cost F is at least as high as the flow benefit v of being satisfied with a house.*

Both low valuation and double owners can enter the housing market as seller. With $v \leq F$, the flow benefits (not necessarily the benefits to search) of low valuation owners exceed those of double owners. Because double owners were low valuation owners before buying a second home, on average double owners have spent more time on the market selling their house than low valuation owners. In the language of Albrecht, Anderson, Smith, and Vroman (2007), it is therefore natural to model double owners as the more desperate sellers, and low valuation owners as the more relaxed sellers. Assumption 1 achieves this, as is shown in the next section.

3 The simultaneous equilibrium

3.1 Equilibrium prices

Because both low valuation owners and renters are buyers in a simultaneous equilibrium, take-it-or-leave-it offers result in both P_L and P_R . The resulting price difference follows from buyers' two different values of changing states. A low valuation owner pays

$$P_L = V_D - V_L, \tag{6}$$

while a renter pays

$$P_R = V_H - V_R. \tag{7}$$

The price a seller receives therefore depends on the buyer type. Assume that the buyer type that pays the lower of the two prices cannot be excluded from the search process of the seller, or that the seller cannot save on search costs by doing so. The question is then whether a seller accepts an immediate sale to a buyer of the type that pays the lower price, or continues to search for a buyer of the type that pays the higher price. When households

accept to sell to buyers that pay the lower price, there is price dispersion in equilibrium. In this case, households that search for a buyer expect to sell at P_L with a probability corresponding to the share of low valuation owners in the population of buyers, and at P_R with a probability corresponding to the fraction of renters. The expected price is thus

$$P_i = \frac{\mu_L}{\mu_B} P_L + \frac{\mu_R}{\mu_B} P_R \quad (8)$$

Subtracting (3) from (5) and rearranging gives P_L as a function of P_R

$$P_L = \frac{v - F + pP_R}{r + p}. \quad (9)$$

With Assumption 1, prices can then be ranked as in Lemma 2.

Lemma 2. *Renters pay a higher price than low valuation owners, and this price is positive.*

Proof. Take-it-or-leave-it offers from sellers make renters indifferent to buy, so that $V_R = 0$ and $P_R = V_H$. Now note that $V_L > 0$, and that this implies that $P_R = V_H > 0$, as can be seen by rewriting (2) to $(r + \eta)V_H = v + u + \eta V_L$.

Assumption 1 and $P_R > 0$ can subsequently be shown to imply that $P_R > P_L$. From (9) it follows that the difference between both prices is given by

$$P_L - P_R = \frac{v - F - rP_R}{r + p}. \quad (10)$$

P_R thus exceeds P_L if and only if $P_R > \frac{v-F}{r}$, so that $P_R > P_L$ if $P_R > 0$ and $v \leq F$.⁸ \square

A buyer always starts as a low valuation household, and becomes a renter in case she sells her old house before buying a new one. Given that the probability that a buyer has become a renter increases over time, the expected price that a buyer will pay, conditional on the time she is looking for a house, rises with time spent searching. The result that renters pay higher prices than low valuation owners illustrates that renters are more desperate buyers

⁸Note that prices are only equal for $P_L = P_R = (v - F)/r$.

than low valuation owners. Renters gain more by transacting and switching states than low valuation owners, so that sellers that can make take-it-or-leave-it offers are able to extract a higher surplus from renters than from low valuation owners. Low valuation owners, on the other hand, will become a double owner by buying a house. Their alternative road towards high valuation ownership is via rental housing. Since double owners have lower flow benefits than low valuation owners (by Assumption 1), low valuation owners will only be willing to make the transition towards high valuation ownership via double ownership if they pay a lower price than renters.

Finally, as derived in Appendix A, the price that renters pay is

$$P_R = \frac{\frac{v+u}{r} + \frac{\eta}{r+p} \frac{u-c}{r} + \frac{\eta p}{(r+p)^2} \frac{\mu_L}{\mu_B} \frac{v-F}{r}}{1 + \frac{\eta}{r+p} + \frac{\eta p}{(r+p)^2} \frac{\mu_L}{\mu_B}}, \quad (11)$$

which can be substituted in (9) to yield P_L . Prices reflect the weighted averages of households' (perpetual) benefits and costs, taking discounting, expected durations, and state transition probabilities into account. These expected durations and hazard rates are the subject of the next subsection.

3.2 Steady state stocks and flows in the housing market

If all equilibrium existence conditions are satisfied so that p and q do reflect transaction rates, then steady state stocks and flows follow mechanically from the model. In steady state, the fractions of high valuation, low valuation and double owners, and renters are constant: $\dot{\mu}_H = 0$, $\dot{\mu}_L = 0$, $\dot{\mu}_D = 0$, and $\dot{\mu}_R = 0$. The following conditions then hold:

$$\dot{\mu}_H = 0 : \quad q\mu_R + p\mu_D = \eta\mu_H, \quad (12)$$

$$\dot{\mu}_R = 0 : \quad p\mu_L = q\mu_R, \quad (13)$$

$$\dot{\mu}_L = 0 : \quad \eta\mu_H = p\mu_L + q\mu_L, \quad (14)$$

$$\dot{\mu}_D = 0 : \quad q\mu_L = p\mu_D. \quad (15)$$

Important for prices, the composition of buyers in steady state can be expressed in terms of p and q from these steady state conditions. Substituting (15) into (12) gives that the number of transactions is equal to

$$q(\mu_R + \mu_L) = q\mu_B = \eta\mu_H = \mu_L(p + q), \quad (16)$$

where the last equality follows from (14). Consequently, the fraction of low valuation owners in the total measure of buyers is given by $\mu_L/\mu_B = q/(p + q)$. From the composition of μ_B it then follows that $\mu_R/\mu_B = p/(p + q)$. Similarly, substituting (15) into (14) and following the same steps, yields $\mu_L/\mu_S = p/(p + q)$, and $\mu_D/\mu_S = q/(p + q)$. Lemma 3 summarizes the impact of the housing supply on the steady state stocks and flows for the simultaneous equilibrium.

Lemma 3. $h \gtrless 1 \Leftrightarrow p \lesseqgtr q \Leftrightarrow \mu_R \lesseqgtr \mu_L \lesseqgtr \mu_D$.

Proof. From (13) and (15) it follows that $\theta = p/q = \mu_R/\mu_L = \mu_L/\mu_D$. Combining this with (1) yields

$$\frac{1 - h}{\mu_L} = \frac{\mu_R}{\mu_L} - \frac{\mu_D}{\mu_L} = \frac{p}{q} - \frac{q}{p} = \frac{p^2 - q^2}{pq}, \quad (17)$$

which implies the (in)equalities to be proven. \square

We see that the rate to sell is smaller than the rate to buy if and only if there are more houses than households, and that this implies that there are more double owners than low valuation owners, which are more numerous than renters. Note that this result on the stocks and flows in the housing market is independent of equilibrium prices, given that all matches are consummated. Maury and Tripier (2010) presents the first equilibrium search model showing the importance of allowing households to choose the order of buying and selling. For simplicity they assume $p = q$, but also in their model this would imply $h = 1$. Unfortunately, as shown in Lemma 1, explicitly modeling a housing supply equal to one is inconsistent with renters to disappear in steady state, which happens if all low valuation owners first buy.

With the steady state conditions, all fractions can be written in terms of μ_L , and using that fractions sum to one, yields

$$\mu_L = \frac{1}{1 + \frac{p}{q} + \frac{p+q}{\eta} + \frac{q}{p}}. \quad (18)$$

The comparative statics are natural. The measure of low valuation owners is increasing in η and decreasing in p and q . From (17), the measure of renters is increasing in p and decreasing in q , whereas the reverse holds for the measure of double owners. The matching function, (17), and (18) give three equations in the three endogenous variables $p > 0$, $q > 0$, and μ_L , for a given housing supply $h \in (0, 2)$ and preference shock $\eta > 0$. Lemma 4 states that unique solutions for p , q , and μ_L result. All other stocks of households then follow uniquely as well. The proof is in Appendix B.

Lemma 4. *The steady state relationships between the stocks and flows in the simultaneous equilibrium (as defined by (17), (18) and matching function $M = m_0 \mu_B^\alpha \mu_S^{1-\alpha}$) have unique positive and real solutions for transaction rates p and q and the measure of low valuation owners μ_L , for a given housing supply $h \in (0, 2)$ and preference shock $\eta > 0$. Moreover, if and only if $h = 1$, the single transaction rate ($p = q$) is equal to m_0 .*

3.3 Participation and equilibrium existence conditions

The existence of the proposed equilibrium requires costly search effort from sellers, and requires high valuation owners not to enter the market. In addition, it requires sellers to sell to low valuation owners, even if the latter pay a lower price than renters. Besides, positive prices are a prerequisite for the plausibility of the proposed equilibrium. Lemma 2 shows that P_R is always positive. Unfortunately, with Assumption 1, the same does not necessarily hold for P_L . However, as proven in Appendix C, Proposition 1 states that if $P_L > 0$, then double owners enter the market as sellers, while high valuation owners never enter the market as sellers. Remember that a double owner is subject to preference shocks again once the house is sold, so that spending costly search effort is not trivial. Moreover,

double owners always sell to both types of buyers. Besides, if a high valuation owner would buy a second house, she would be made indifferent. I simply assume this transaction never happens in equilibrium.⁹

Proposition 1. *In the simultaneous equilibrium with take-it-or-leave-it offers from sellers,*

1. *double owners sell to both renters and low valuation owners;*
2. *high valuation owners do not enter the market as sellers; and*
3. *if the price that low valuation owners pay is positive, then double owners search.*

Although double owners enter the market as sellers (if $P_L > 0$) and sell to both types of buyers, low valuation owners are more picky, as argued in Proposition 2. The proof is in Appendix C.

Proposition 2. *Double owners are more desperate sellers than low valuation owners in the sense that*

1. *if low valuation owners search, double owners do so as well, but not necessarily the other way around;*
2. *if low valuation owners sell to both types of buyers, double owners do so as well, but not necessarily the other way around.*

Since double owners have lower flow benefits than low valuation owners, this result is similar to Albrecht, Anderson, Smith, and Vroman's results on the matching pattern of buyers and sellers. If relaxed owners sell to their non-preferred type of buyer, then desperate owners do so as well. As a result, strict negative assortative matching is ruled out. Albrecht, Anderson, Smith, and Vroman (2007) elaborate on the possible equilibrium matching patterns of their model, while I focus on the conditions under which households sell to both types of buyers. By focusing on the participation constraints to search, equilibrium multiplicity arises from a different source than in Albrecht, Anderson, Smith, and

⁹Only a double owner would be willing to sell to a high valuation owner, but this transaction has no surplus. A transaction between a high and a low valuation owner has a negative surplus.

Vroman. If the participation constraint of low valuation owners is not satisfied, while it is for double owners, then an equilibrium in which low valuation owners always first buy and then sell might be possible. The next section shows such a sequential equilibrium.

4 The ‘first-buy-then-sell’ equilibrium

In a sequential equilibrium low valuation owners do not enter the market as sellers, while for the market not to break down, double owners must continue to do so. In principle this difference in behavior is possible, because double owners are more desperate to sell than low valuation owners by Proposition 2.¹⁰ Besides, we know that if low valuation owners stop entering the market as seller, renters disappear in steady state. Allowing renters to disappear requires a housing supply $h \in (1, 2)$ by Lemma 1, which is therefore assumed in this section. The transaction rates, household measures, asset values of household types, and thus prices, generally take different values in a sequential equilibrium than in a simultaneous equilibrium. I use primes to denote these variables in the sequential equilibrium.

The single price in the market results from transactions between low valuation and double owners, the only buyers and sellers respectively. When sellers make take-it-or-leave-it offers and low valuation owners only search as buyers, the asset value of being a low valuation owner is simply u/r . The price makes low valuation owners indifferent to buy, and is thus $P'_L = V'_D - u/r$. Solving (2), (3), and (5) under these assumptions, yields

$$P'_L = \frac{v - F - c}{r} + \frac{p'}{r} \frac{v}{r + \eta}. \quad (19)$$

The difference between the value for a high valuation owner relative to a low valuation owner, $V'_H - V'_L$, is $v/(r + \eta)$. The price P'_L therefore reflects the perpetual flows of double ownership minus those of low valuation ownership ($v - F - c$), plus the option value of selling the old house and enjoying the benefits of a high relative to a low valuation owner.

¹⁰Note from a contraposition of the proof of this proposition that if $P_L > P_R$, low valuation owners always search if double owners search.

The plausibility of the equilibrium requires a nonnegative value for P'_L , but this is only the case if $p' \geq \underline{p}$, with

$$\underline{p} \equiv \frac{F + c - v}{r} \bigg/ \frac{v}{r(r + \eta)}, \quad (20)$$

which is always positive under Assumption 1. If P'_L is indeed nonnegative, then double owners search, as stated in Lemma 5 and proven in Appendix C. Moreover, high valuation owners never enter the market as sellers if low valuation owners do not do so, which defines the sequential equilibrium.

Lemma 5. *In a sequential equilibrium with take-it-or-leave-it offers from sellers,*

1. *high valuation owners do not search if low valuation owners do not search; and*
2. *if the price is nonnegative, then double owners search.*

Now imagine a single low valuation owner that considers to deviate from the sequential search strategy by simultaneously searching as a seller. A single household will only be able to sell her house to other low valuation owners, making one of them a take-it-or-leave-it offer. For the sequential equilibrium to exist, it must not be worthwhile to do so. The threshold for a low valuation owner to enter the market as seller is when benefits equal costs, thus if

$$p'(V'_R - V'_L + P'_L) = c$$

Assume for now that double owners sell to renters. The low valuation owner then knows that once she becomes a renter, she will be made a take-it-or-leave-it offer $P'_R = V'_H$ by a double owner and her value will be zero. Since the single deviating household has measure zero, the presence of a single renter has no impact on the price that double owners expect to receive for their old house, so that their participation constraint is unaffected. If the low valuation owner finds a new house before being able to sell her old house, she will also be made a take-it-or-leave-it offer. For that reason, if the participation constraint to enter the market as seller is satisfied with equality, the asset value of a low valuation owner is u/r , just as in the sequential equilibrium. As a result, in this case the price a

deviating household will be able to ask for her house is equal to the going price in the market: P'_L . In addition, V'_H will also be the same as in the sequential equilibrium, so that $P'_R = v/(r+\eta) + u/r$. Because Lemma 2 can be checked to apply in a sequential equilibrium in which low valuation owners are indifferent to enter the market as sellers, $P'_R > P'_L$ and double owners indeed sell to renters.¹¹ Under these conditions, a low valuation owner is indifferent to enter the market as seller for

$$p'(P'_L - V'_L) = p' \frac{v - F - c - u}{r} + \frac{p'^2}{r} \frac{v}{r + \eta} = c \quad (21)$$

Lemma 6 states that there exists a unique positive p' for which this participation constraint is satisfied with equality.

Lemma 6. *If the sequential equilibrium exists, there exists a unique positive \bar{p} such that a single low valuation owner is indifferent to enter the market as seller.*

Proof. The solutions of (21) are given by

$$p' = \frac{\frac{F+c+u-v}{r} \pm \sqrt{\left(\frac{v-c-F-u}{r}\right)^2 + \frac{4c}{r} \frac{v}{r+\eta}}}{\frac{2}{r} \frac{v}{r+\eta}}$$

Since the determinant is positive, two real solutions exist. Since the absolute value of the term in parentheses is the same as the first term, the square root always dominates the first term. As a result, one of the two solutions is positive, \bar{p} , while the other is negative. \square

Proposition 3 gives the range of selling probabilities for which the sequential equilibrium always exists, in which \bar{p} is the unique threshold for a low valuation owner not to enter the market as seller in the sequential equilibrium.

Proposition 3. *For any $p' \in [\underline{p}, \bar{p}]$, which is nonempty, the sequential equilibrium exists with a nonnegative price P'_L . For $p' > \bar{p}$ no sequential equilibrium can exist.*

¹¹Subtract (3) from (5) without substituting $p(V_R - V_L + P_i) = c$ and rearrange to note that $V_D - V_L < V_H$ under Assumption 1.

Proof. Lemma 6 states that there is a unique \bar{p} for which a low valuation owner is indifferent to enter the market as seller in the sequential equilibrium. The shape of the parabola (21), in particular $\frac{v}{r(r+\eta)} > 0$, then implies that for $0 < p' < \bar{p}$, the net benefit of search is negative. Because of the max-function, the participation constraint is the same in case low valuation owners are not indifferent but have negative net benefits from search, so that low valuation owners do not search for $0 < p' < \bar{p}$. Using Lemma 5, all equilibrium existence conditions are then satisfied if parameters are such that $P'_L \geq 0$ for some $p' \in (0, \bar{p}]$. We know that this is the case for $p' > \underline{p}$. Since both the first term and the square root of \bar{p} are larger than the numerator of \underline{p} , $\underline{p} < \bar{p}$. As a result, for any $p' \in [\underline{p}, \bar{p}]$ the sequential equilibrium exists. For $p' > \bar{p}$ low valuation owners have incentives to search, so that no sequential equilibrium can exist. \square

Finally, p' follows uniquely from the stocks and flows in a sequential equilibrium, as stated in Lemma 7. The proof is in Appendix B.

Lemma 7. *The steady state relationships between the stocks and flows in the sequential equilibrium have unique positive solutions for transaction rates p' and q' and the measures μ'_L , μ'_D , and μ'_H for a given housing supply $h \in (1, 2)$ and preference shock $\eta > 0$.*

In the next section I investigate whether a simultaneous equilibrium can exist for the same fundamentals as a sequential equilibrium.

5 Multiple equilibria in the housing market

5.1 Desperate buyers complementarities

We have seen that there is a unique threshold \bar{p} for a low valuation owner not to enter the market as seller in the sequential equilibrium. In this subsection I study the same threshold for the simultaneous equilibrium. Consider a simultaneous equilibrium where the net benefit of low valuation owners to search to sell is exactly zero, so that $V_L = u/r$.

In this case, P_L is simply $V_D - u/r$, and is given by

$$P_L = \frac{v - c - F}{r} + \frac{p}{r} \frac{v}{r + \eta} + \frac{p \mu_R}{r \mu_B} (P_R - P_L). \quad (22)$$

Comparing (19) and (22), the difference between P_L and P'_L is given by the additional term $\frac{p \mu_R}{r \mu_B} (P_R - P_L)$. Consequently, assuming for now that the rate to sell is the same across equilibria ($p = p'$), the ranking of P_R and P_L in the *simultaneous* equilibrium determines the ranking of P_L and P'_L *across* equilibria. Because $P_R > P_L$ by Lemma 2 (which still applies if $V_L = u/r$), P'_L in the sequential equilibrium is smaller than P_L in the simultaneous equilibrium at $V'_L = V_L = u/r$. As a result, with $P_R > P_L > P'_L$ the benefits of selling at $V'_L = V_L = u/r$ are higher in the simultaneous equilibrium than in the sequential equilibrium. The presence of more desperate buyers that pay higher prices (renters) gives other households incentives to sell and thus to become such desperate buyers as well. There is a complementarity in the order of buying and selling. As claimed in Proposition 4, low valuation owners in the simultaneous equilibrium may therefore search for a p^* for which low valuation owners in the sequential equilibrium do not search.

Proposition 4. *There exists a $p^* \in (0, \bar{p})$ for which low valuation owners in the simultaneous equilibrium are indifferent to enter the market as sellers or not, while low valuation owners in the sequential equilibrium have negative net benefits from entering the market as sellers, if these equilibria exist. Moreover, $u \geq F$ is sufficient for $p^* > \underline{p}$, and thus for the sequential equilibrium to exist.*

Proof. Because $V_L = u/r$ if low valuation owners are indifferent, in the simultaneous equilibrium $P_R = V_H = v/(r + \eta) + u/r$. Rewriting (9) as $P_R - P_L = (rP_L - v + F)/r$, substituting this into (22) and rearranging, yields

$$\frac{\mu_L}{\mu_B} P_L = \frac{\mu_L}{\mu_B} \frac{v - F}{r} - \frac{c}{r} + \frac{p}{r} \frac{v}{r + \eta}.$$

Low valuation owners in the simultaneous equilibrium are thus indifferent to search if

$$p \left(\frac{\mu_R}{\mu_B} P_R + \frac{\mu_L}{\mu_B} P_L - V_L \right) = p \left(\frac{\mu_L}{\mu_B} \frac{v - F - u}{r} - \frac{c}{r} + \frac{\mu_R}{\mu_B} \frac{v}{r + \eta} \right) + \frac{p^2}{r} \frac{v}{r + \eta} = c.$$

For the same reasons as in Lemma 6, this quadratic equation has one positive solution as well: p^* . Moreover, $p^* < \bar{p}$ because the term in parentheses is larger than $(v - F - c - u)/r$ in (21), while the remainder of the equation is the same. Besides, the sequential equilibrium exists with a positive price at p^* if $p^* > \underline{p}$, thus if

$$\begin{aligned} \frac{\mu_L}{\mu_B} \frac{F + u - v}{r} + \frac{c}{r} - \frac{\mu_R}{\mu_B} \frac{v}{r + \eta} &> \frac{F + c - v}{r} \\ \Leftrightarrow \frac{\mu_L}{\mu_B} (r + \eta) u &> \frac{\mu_R}{\mu_B} ((r + \eta) F - \eta v) \end{aligned}$$

By Lemma 3, if $h \in (1, 2)$ then $\mu_R < \mu_L$, so that $u \geq F$ is sufficient for $p^* > \underline{p}$. \square

Consequently, there exists a selling rate p^* for which the participation constraint of low valuation owners in the simultaneous equilibrium is satisfied, while it is not in the sequential equilibrium. Moreover, the existence of the sequential equilibrium with a positive price at p^* is guaranteed if $u \geq F$ (which seems plausible given that being homeless is clearly very undesirable). It is in this sense that Maury and Tripier (2010) speak about the existence of ‘multiple equilibria’. However, p and p' are endogenous. Proposition 5 shows that the rate to sell is smaller in the simultaneous steady state than in the sequential steady state. Lemma 8 is instrumental to this result.

Lemma 8. *Comparing the sequential and the simultaneous steady state for the same η , the steady state with the larger number of transactions is the steady state with the higher measure of high valuation owners.*

Proof. In both steady states the measure of high valuation owners is the number of transactions divided by η , as can be seen in (16) for the simultaneous equilibrium and in (27) for the sequential equilibrium. Consequently, if and only if the number of transactions is larger, there are more high valuation owners in steady state. \square

One might think that the simultaneous equilibrium, with two types of sellers and two types of buyers, features more sellers *and* more buyers than the sequential equilibrium. As a result, since the matching function is increasing in its arguments, the simultaneous equilibrium would be characterized by more transactions, and by Lemma 8, by a larger measure of high valuation owners. However, the next subsection shows by example that there are not necessarily more buyers in the simultaneous equilibrium than in the sequential equilibrium. On the other hand, it does follow from Proposition 5 that there are always more sellers in the simultaneous equilibrium. In addition, the time to sell is always longer in this equilibrium than in the sequential equilibrium.

Proposition 5. *For the same fundamentals η and $h \in (1, 2)$, there are fewer double owners and more low valuation owners, and the rate to sell is larger and the rate to buy is smaller in the sequential steady state than in the simultaneous steady state, if these equilibria coexist.*

Proof. Since renters disappear in the sequential steady state, from (1) it follows that for the same $h \in (1, 2)$ there must be fewer double owners in the sequential steady state than in the simultaneous steady state. Now suppose that there would also be more low valuation owners in the simultaneous steady state (next to more double owners and renters). In this case there would be more sellers *and* more buyers in the simultaneous steady state. Because the matching function is increasing in its arguments, the number of transactions would be higher, and by Lemma 8 there would also be more high valuation owners for the same η . However, not all fractions can be larger. Consequently, there must be fewer low valuation owners in the simultaneous steady state than in the sequential steady state.

Besides, in both steady states market tightness is given by $\frac{\mu_L}{\mu_D} = \frac{p}{q}$. Note that p and q (and p' and q') move in opposite directions due to the constant returns to scale matching function, and remember from Lemmas 4 and 7 respectively that p and q , and p' and q' are unique. As a result, $p' > p$ and $q' < q$, if both equilibria exist for the same fundamentals. \square

The fact that p' in the sequential equilibrium is endogenously larger than p in the simultaneous equilibrium has two important consequences. First, since P'_L and P_L are

increasing in p' and p respectively, this opens up the possibility that P'_L exceeds P_L . In this case, for the net benefit of search to be nonnegative in the simultaneous equilibrium and negative in the sequential equilibrium, P_R must exceed P_L even more than if the rate to sell would not change across equilibria.

Secondly, even if prices are such that a selling rate p^* or a range of selling rates exists for which the participation constraint in the simultaneous equilibrium is satisfied while it is not in the sequential equilibrium, then still this range of selling rates may not be consistent with a range of fundamentals for h and η . The issue is illustrated in figure 2. Equilibrium existence conditions of the sequential equilibrium may be satisfied for $p' \in (\underline{p}, \bar{p})$, and those of the simultaneous equilibrium for any p larger than $p^* \in (\underline{p}, \bar{p})$ (the shaded areas). However, because $p < p'$ there may be no fundamentals that make both p and p' lie within (p^*, \bar{p}) . In contrast, the figure shows a situation in which multiple equilibria do exist.

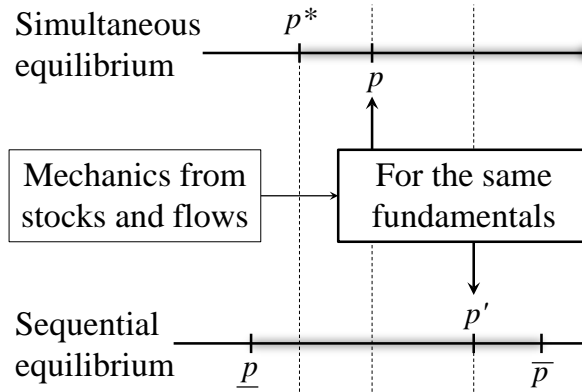


Figure 2: Existence conditions for multiple equilibria

It thus follows from Proposition 5 that the simultaneous equilibrium is ‘stable’ in the sense that if low valuation owners stop entering the market as seller, the rate at which sellers can sell their house increases, (partly) restoring incentives to search. On the other hand, the rate at which buyers can buy a new house decreases. As a result, if other low valuation owners stop entering the market as seller, households that continue to enter the market as sellers and become renters, are likely to be stuck in temporary housing for a longer duration. However, since renters are made a take-it-or-leave-it offer, sellers make

renters indifferent to their time on the market. Under these assumptions, at least with a constant returns to scale matching function, equilibrium multiplicity can therefore never arise from search externalities per se. The source of equilibrium multiplicity must therefore lie in price dispersion and the composition of buyer types.

In the next subsection I show numerically that there exists a housing supply and a preference shock resulting in a p and a p' , such that at p low valuation owners enter the market as sellers, while $p' \in (\underline{p}, \bar{p})$ so that the sequential equilibrium exists with a positive price. It follows from Propositions 1 and 2 that the single remaining condition for the simultaneous equilibrium to exist is that low valuation owners sell to both types of buyers. I check that this equilibrium existence condition is also satisfied at the same p . As a result, multiple equilibria exist for the same housing supply and preference shocks. The next subsection also compares these equilibria in terms of welfare.

5.2 Two numerical examples and welfare

In this subsection I give two numerical examples, each resulting in multiple equilibria. The examples differ in the ranking of the number of transactions across equilibria. It follows from Lemma 8 that these examples then also differ in their ranking of the measure of high valuation owners across equilibria. Comparing the examples in terms of welfare, however, the sequential equilibrium dominates the simultaneous equilibrium in both cases. This numerical result is representative for most parameter values experimented with for which multiple equilibria exist.

I define welfare Ω as the sum of the asset values of each household type, weighted for their share in the population. In addition, I include the housing services enjoyed by renters as a perpetuity, also weighted by their fraction, effectively assuming that all households are also landlords and share in the rental payments. An example in which the sequential equilibrium dominates the simultaneous equilibrium, even if the measure of high valuation owners is larger in the simultaneous equilibrium, then exists due to a larger value for double

owners in the sequential equilibrium, as claimed in Proposition 6.

Proposition 6. *If welfare is larger in the sequential equilibrium than in the simultaneous equilibrium, if $P_L, P'_L > 0$, and if there are more high valuation owners in the simultaneous equilibrium than in the sequential equilibrium, then the value of double ownership is larger in the sequential equilibrium than in the simultaneous equilibrium.*

Proof. Remember that $V_L \geq V'_L$ due to the net benefits of search, so that also $V_H \geq V'_H$. We also have that $V'_H > V'_L$ (as noted in the proof of Lemma 5 in Appendix C) and that $V_H > V_L$ (as follows from Appendix A, given that $P_R > P_L$). In addition, for positive prices by definition $V_D > V_L$ and $V'_D > V'_L$. Moreover, the perpetuity of rental payments $(v+u)/r$ exceeds $V'_H = v/(r+\eta) + u/r$, and V_H (which is P_R , in which $(v+u)/r$ is the largest of the terms of the weighted average making up P_R). Finally, by Proposition 5 there are more renters and double owners, and fewer low valuation owners in the simultaneous than in the sequential equilibrium. The proposition can then be proven by showing that if $V_D > V'_D$ and $\mu_H > \mu'_H$, then $\Omega > \Omega'$. If $V_D > V'_D$, then

$$\mu'_D V_D + \mu'_L V_L + \mu'_H V_H > \mu'_D V'_D + \mu'_L V'_L + \mu'_H V'_H = \Omega'.$$

Now subtracting the left-hand side above from the definition of Ω , yields

$$(\mu_D - \mu'_D)V_D + (\mu_L - \mu'_L)V_L + (\mu_H - \mu'_H)V_H + \mu_R \left(\frac{v+u}{r} \right).$$

If $\mu_H > \mu'_H$ (as in the numerical example), then this difference in welfare is positive, because only $\mu_L - \mu'_L$ is negative, while $V_L < V_H < (v+u)/r$, $V_L < V_D$, and $\mu_D - \mu'_D + \mu_L - \mu'_L + \mu_H - \mu'_H + \mu_R = 0$. Consequently, $\Omega > \Omega'$. \square

So even if there are slightly more high valuation owners and fewer low valuation owners in the simultaneous equilibrium, and even though their asset values are higher in the simultaneous equilibrium, the sequential equilibrium is often superior in terms of welfare. If all of this is the case, as in the next numerical example, it is because the asset value

for double owners is larger in the sequential equilibrium. As a corollary, note from (5) that the ranking of V_D and V'_D must result from the fact that $p' > p$, given that $V_H > V'_H$ and that average prices are larger in the simultaneous equilibrium. For that reason, the endogenously higher selling rate of the sequential equilibrium can improve welfare, because the risk of paying double housing expenses for a long time is smaller in a housing market in which all households take this risk. Maury and Tripier (2010) take the asset value of a high valuation owner to represent total welfare. Unfortunately, given the result above, their procedure can give wrong conclusions if one allows transaction rates to be endogenous.

In my numerical examples, I calibrate all flows at a quarterly frequency, and for that reason set discount rate r at 0.012. Assume that households want to move on average once in ten years between owner-occupied houses, and therefore set $\eta = 0.025$. I normalize the flow utility u from not being homeless at 5 (thousand). Following Assumption 1, I assume that $v > F$ and set additional housing services from being satisfied at 2 and double housing expenses at 2.5. Seller's search costs c are 0.1. In the first example, the Cobb-Douglas matching function has parameters $m_0 = 1$ and $\alpha = 0.5$, so that buyers and sellers are given an equal elasticity in the matching process.

I use (21) to find the selling rate \bar{p} for which low valuation owners in the sequential equilibrium are indifferent to search, and back out the housing supply h that results in this \bar{p} given the shock process η . This procedure results in $h \approx 1.19$ for the parameters chosen above. Subsequently I experiment with a slightly larger housing supply, and check whether both the simultaneous and the sequential equilibrium exist for the same h and η .

The endogenous variables (now without primes) of the first numerical example are presented in table 2. For the chosen parameters, $P_R > P'_L > P_L > 0$, and the equilibrium existence conditions for both equilibria are satisfied. In particular, low valuation owners enter the market as sellers and sell to both types of buyers in the simultaneous equilibrium, while they do not search in the sequential equilibrium. It follows that average prices are higher in the simultaneous equilibrium. However, since Lemma 3 shows that $\mu_R < \mu_L < \mu_D$ in the simultaneous equilibrium for $h > 1$ (as confirmed in the numerical example), median

and modal prices are higher in the sequential equilibrium. The numerical example also confirms the results of Proposition 5 that μ_D and q are larger in the simultaneous equilibrium, and μ_L and p are larger in the sequential equilibrium. We see that in this first numerical example there are more transactions and more high valuation owners in the simultaneous equilibrium than in the sequential equilibrium, but these differences are tiny. Moreover, note that the measure of buyers in the simultaneous equilibrium ($\mu_L + \mu_R$) is smaller than in the sequential equilibrium (μ_L). Finally, the sequential equilibrium is superior to the simultaneous equilibrium in terms of welfare.

Variable	Simultaneous equilibrium	Sequential equilibrium
P_L	417.1	417.3
P_R	470.7	
p	0.103	0.104
q	9.741	9.639
μ_L	0.00204	0.00209
μ_D	0.19380	0.19377
μ_R	0.00002	0
μ_H	0.8041413	0.8041409
# of transactions	0.02010353	0.02010352
Welfare	540.99	541.00

Table 2: Endogenous variables of a simultaneous and a sequential equilibrium for the same fundamentals, in particular $\alpha = 0.5$ and $h = 1.19377$.

Because the measure of buyers in the simultaneous equilibrium can be larger than in the sequential equilibrium, as in the numerical example of table 2, the simultaneous equilibrium is not necessarily characterized by more transactions and a larger measure of high valuation owners. Simply by increasing the elasticity of the matching function, the number of transactions and the measure of high valuation owners can be larger in the sequential equilibrium than in the simultaneous equilibrium. Table 3 presents a second numerical example with α equal to 0.55, a reconstructed housing supply still approximately 1.19 (using the procedure explained above), while keeping all other parameters fixed. The first seven rows show no qualitative differences, but now the number of transactions and the measure of high valuation owners is larger in the sequential equilibrium than in the simul-

taneous equilibrium. Welfare is still larger in the sequential equilibrium, but now also its larger measure of high valuation owners contributes to its superiority over the simultaneous equilibrium, and not only its smaller time to sale.

Variable	Simultaneous equilibrium	Sequential equilibrium
P_L	416.9	417.6
P_R	470.8	
p	0.102	0.104
q	6.465	6.381
μ_L	0.00306	0.00315
μ_D	0.19353	0.19348
μ_R	0.00005	0
μ_H	0.80337	0.80338
# of transactions	0.0200842	0.0200844
Welfare	540.87	540.89

Table 3: Endogenous variables of a simultaneous and a sequential equilibrium for the same fundamentals, in particular $\alpha = 0.55$ and $h = 1.19348$.

Because p can be interpreted as the rate of liquidity, as it captures the speed at which households can transfer their illiquid assets to cash, the multiple equilibria can represent phases of high and low liquidity. The period before the Great Recession can be represented by the sequential equilibrium, in which liquidity is high and relatively many low valuation owners buy a new house before selling their old houses, just as in table 1. The period during the Great Recession can then be represented by the simultaneous equilibrium. In a sequential equilibrium the steady state fraction of people paying double housing expenses is smaller than in a simultaneous equilibrium, due to a smaller time to sale. This matches with the common observation that the problem of being stuck with two houses is more severe during than it was before the Great Recession. Finally, as reported in table 1, the number of transactions is substantially larger before than during the Great Recession. The numerical examples show that this experience can be consistent with the multiple equilibria presented in this paper if the elasticity of the matching function is sufficiently large.

6 Conclusion

Even with a constant returns to scale matching function, an equilibrium search model of the housing market may be characterized by multiple equilibria. Allowing households to choose the order of buying and selling is likely to result in more or less desperate households. More desperate buyers accept to pay higher prices, giving rise to endogenous price dispersion. The presence of these desperate buyers motivates sellers to search and to become desperate buyers themselves. However, in the absence of these buyers, sellers would not search and would not become desperate buyers. Indeed, when households first buy a new house before selling their old house, in steady state nobody rents temporary housing. However, renters pay higher prices than households looking for a second house. For that reason, when households stop to enter the market as sellers, households paying relatively high prices disappear from the market. As a result, it is individually rational to stop searching as a seller. However, if households would continue to enter the market as sellers, renters would survive in the market, and their willingness to pay higher prices would have made entering the market as sellers rational. The same mechanism applies if all household first sell and households owning two houses disappear in steady state.

Interestingly, in the equilibrium in which all households first buy a new house and thus become the owner of two houses, the steady state fraction of owners of two houses is smaller than in the equilibrium in which some households first sell their house and only then buy a new house, never owning two houses. The reason is that, although indeed in the 'first-buy-then-sell' equilibrium the inflow rate into the state of owning two houses is larger than in the latter equilibrium, the outflow rate is so much larger in the 'first-buy-then-sell' equilibrium that the steady state fraction of households owning two houses is lower than in the other equilibrium. Consequently, the risk of paying double housing expenses for a long time is smaller in a housing market in which all households take this risk, as compared to a market where some households avoid this risk by first selling their old house, only buying a new house afterwards and renting in the meantime. Since the outflow rate out of the state

of two houses is the inverse of the time to sale, it captures the rate at which an illiquid asset can be transformed in cash. It can therefore be argued that the 'first-buy-then-sell' equilibrium is characterized by a higher liquidity. I show numerically that this effect can make the 'first-buy-then-sell' equilibrium superior in terms of welfare.

Phases of high and low liquidity may therefore be the result of different self-fulfilling expectations about the search behavior of market participants. The 'first-buy-then-sell' equilibrium, with its higher liquidity and smaller fraction of double owners, can represent the period before the Great Recession, whereas the equilibrium in which households enter the market as buyers and sellers simultaneously can represent the period of the Great Recession. If the contribution of buyers towards the matching process is sufficiently large, these two equilibria can also explain the difference in the number of transactions. This paper, however, does not consider the dynamics between the steady states. Future research might clarify the transition paths between different steady states, and study their stability.

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Appendices

A Derivation of the price paid by renters

Take-it-or-leave-it offers from sellers imply that low valuation owners have zero benefits from entering the market as buyer. The difference between (2) and (3) is then

$$\begin{aligned}
r(V_H - V_L) &= v - \eta(V_H - V_L) + c - p \left(V_R - V_L + \frac{\mu_L}{\mu_B} P_L + \frac{\mu_R}{\mu_B} P_R \right) \\
\Leftrightarrow (r + \eta + p)(V_H - V_L) &= v + c - p \left(V_R - V_H + \frac{\mu_L}{\mu_B} P_L + \frac{\mu_R}{\mu_B} P_R \right) \\
&= v + c + p \left(P_R - \frac{\mu_L}{\mu_B} P_L - \frac{\mu_R}{\mu_B} P_R \right) \\
&= v + c + p \frac{\mu_L}{\mu_B} (P_R - P_L) \\
\Leftrightarrow V_H - V_L &= \frac{v + c}{r + \eta + p} + \frac{p}{r + \eta + p} \frac{\mu_L}{\mu_B} \frac{r P_R - v + F}{r + p}, \tag{23}
\end{aligned}$$

where the last step substitutes (10) for the price difference.

From (7) and (4) it follows that $V_R = 0$ and thus that $P_R = V_H$. Substituting (23) in

(2) gives P_R as a function of the parameters

$$\begin{aligned}
rP_R = rV_H = v + u - \frac{\eta(v+c)}{r+\eta+p} - \frac{\eta p}{r+p} \frac{\mu_L}{\mu_B} \frac{rP_R - v + F}{r+\eta+p} \\
\left(r + \frac{r}{r+p} \frac{\eta p}{r+\eta+p} \frac{\mu_L}{\mu_B} \right) P_R = v + u - \frac{\eta(v+c)}{r+\eta+p} + \frac{\eta p}{r+p} \frac{\mu_L}{\mu_B} \frac{v-F}{r+\eta+p} \\
P_R = \frac{v+u - \frac{\eta(v+c)}{r+\eta+p} + \frac{\eta p}{r+p} \frac{\mu_L}{\mu_B} \frac{v-F}{r+\eta+p}}{r + \frac{r}{r+p} \frac{\eta p}{r+\eta+p} \frac{\mu_L}{\mu_B}}. \tag{24}
\end{aligned}$$

Rearranging gives the price as a weighted average as in 11 in the text.

B Proof of unique p and q

Proof of Lemma 4. From the matching function, $p = m_0(p/q)^\alpha$, so that if $p = q$, then $p = q = m_0$. We know from Lemma 3 that this occurs if and only if $h = 1$. To see that also for $h \neq 1$ there exists a unique and positive solution, combine (17) and (18) to obtain

$$\frac{p^2 - q^2}{pq} = (1-h) \left(1 + \frac{p}{q} + \frac{p+q}{\eta} + \frac{q}{p} \right). \tag{25}$$

Now substitute $q = m_0^{\frac{1}{\alpha}} p^{\frac{\alpha-1}{\alpha}}$ into (25) and rearrange to

$$\frac{1-h}{\eta} \left(\eta + p + m_0^{\frac{1}{\alpha}} p^{\frac{\alpha-1}{\alpha}} \right) + (2-h) \left(\frac{m_0}{p} \right)^{\frac{1}{\alpha}} - h \left(\frac{p}{m_0} \right)^{\frac{1}{\alpha}} = 0. \tag{26}$$

Distinguish between $0 < h < 1$ and $1 < h < 2$:

- First consider the case in which $h \in (1, 2)$. Multiply (26) by $p^{\frac{1-\alpha}{\alpha}}$, and define the left-hand side as a new function:

$$f(p) = \frac{1-h}{\eta} \left(\eta p^{\frac{1-\alpha}{\alpha}} + p^{\frac{1}{\alpha}} + m_0^{\frac{1}{\alpha}} \right) + (2-h) \frac{m_0^{\frac{1}{\alpha}}}{p} - h \frac{p^{\frac{2-\alpha}{\alpha}}}{m_0^{\frac{1}{\alpha}}}$$

I will proof that the function $f(p)$ has only one real and positive solution for which

it is zero. The derivative of this function is

$$f'(p) = \frac{1-h}{\eta} \left(\eta \left(\frac{1-\alpha}{\alpha} \right) p^{\frac{1-2\alpha}{\alpha}} + \frac{1}{\alpha} p^{\frac{1-\alpha}{\alpha}} \right) - (2-h) \frac{m_0^{\frac{1}{\alpha}}}{p^2} - h \left(\frac{2-\alpha}{\alpha} \right) \frac{p^{\frac{1-\alpha}{\alpha}}}{m_0^{\frac{1}{\alpha}}},$$

which is monotonically decreasing given that $0 < \alpha < 1$, $1 < h < 2$, $\eta > 0$, and $m_0 > 0$. Consequently, there is at most one real solution for which $f(p) = 0$. By taking the limits of $f(p)$ as p goes to zero and to infinity, I show that exactly one real solution exists, and that it is positive:

$$\begin{aligned} \lim_{p \rightarrow 0} \left[\frac{1-h}{\eta} \left(\eta p^{\frac{1-\alpha}{\alpha}} + p^{\frac{1}{\alpha}} + m_0^{\frac{1}{\alpha}} \right) + (2-h) \frac{m_0^{\frac{1}{\alpha}}}{p} - h \frac{p^{\frac{2-\alpha}{\alpha}}}{m_0^{\frac{1}{\alpha}}} \right] &= \infty, \\ \lim_{p \rightarrow \infty} \left[\frac{1-h}{\eta} \left(\eta p^{\frac{1-\alpha}{\alpha}} + p^{\frac{1}{\alpha}} + m_0^{\frac{1}{\alpha}} \right) + (2-h) \frac{m_0^{\frac{1}{\alpha}}}{p} - h \frac{p^{\frac{2-\alpha}{\alpha}}}{m_0^{\frac{1}{\alpha}}} \right] &= -\infty. \end{aligned}$$

Since the function $f(p)$ is continuous and goes from infinity to minus infinity for p going from zero to infinity, it has to become zero for a positive p , only once as we know. A unique and positive q follows from the matching function.

- Now consider the case in which $h \in (0, 1)$. Multiply (26) by $p^{-\frac{1}{\alpha}}$, and define the left-hand side as a new function:

$$g(p) = \frac{1-h}{\eta} \left(\frac{\eta}{p^{\frac{1}{\alpha}}} + p^{\frac{\alpha-1}{\alpha}} + m_0^{\frac{1}{\alpha}} p^{\frac{\alpha-2}{\alpha}} \right) + (2-h) \frac{m_0}{p^{\frac{2}{\alpha}}} - \frac{h}{m_0^{\frac{1}{\alpha}}}.$$

Also the continuous function $g(p)$ has only one real and positive solution for which it is zero. Its derivative is

$$g'(p) = \frac{1-h}{\eta} \left(-\frac{1}{\alpha} \frac{\eta}{p^{\frac{1+\alpha}{\alpha}}} + \left(\frac{\alpha-1}{\alpha} \right) p^{-\frac{1}{\alpha}} + \left(\frac{\alpha-2}{\alpha} \right) \frac{m_0^{\frac{1}{\alpha}}}{p^{\frac{2}{\alpha}}} \right) - (2-h) \frac{2}{\alpha} \frac{m_0}{p^{\frac{2+\alpha}{\alpha}}},$$

which is again monotonically decreasing. The limits of $g(p)$ are given by

$$\begin{aligned}\lim_{p \rightarrow 0} \left[\frac{1-h}{\eta} \left(\frac{\eta}{p^{\frac{1}{\alpha}}} + p^{\frac{\alpha-1}{\alpha}} + m_0^{\frac{1}{\alpha}} p^{\frac{\alpha-2}{\alpha}} \right) + (2-h) \frac{m_0}{p^{\frac{2}{\alpha}}} - \frac{h}{m_0^{\frac{1}{\alpha}}} \right] &= \infty, \\ \lim_{p \rightarrow \infty} \left[\frac{1-h}{\eta} \left(\frac{\eta}{p^{\frac{1}{\alpha}}} + p^{\frac{\alpha-1}{\alpha}} + m_0^{\frac{1}{\alpha}} p^{\frac{\alpha-2}{\alpha}} \right) + (2-h) \frac{m_0}{p^{\frac{2}{\alpha}}} - \frac{h}{m_0^{\frac{1}{\alpha}}} \right] &= -\frac{h}{m_0^{\frac{1}{\alpha}}}.\end{aligned}$$

As a result, also for $h \in (0, 1)$ unique positive and real solutions for p and q exist.

Unique positive and real solutions for p and q can be substituted in (18) to obtain a unique and positive μ_L .

Proof of Lemma 7. In a sequential equilibrium there are no renters, and double owners are the only sellers. From (1) with $h \in (1, 2)$, it then follows that $h - 1 = \mu'_D = \mu'_S$. The matching function can subsequently be used to solve for μ'_L , since for a given housing supply and shock process steady state accounting yields

$$\eta(2 - h - \mu'_L) = \eta\mu'_H = m(\mu'_B, \mu'_S) = m(\mu'_L, h - 1). \quad (27)$$

Because the left-hand side is decreasing in μ'_L , while the right-hand side is increasing, a solution for μ'_L is unique. Existence follows from $h \in (1, 2)$ and the fact that $m(0, h - 1) = 0$, combined with continuity and the monotonicity referred to earlier. Given the definition of tightness and the equality of buying and selling transactions, $\theta' \equiv \mu'_B/\mu'_S = \mu'_L/\mu'_D = p'/q'$. Given tightness, housing supply and preference shocks, I can solve for the unique p' (and thus q') by expressing all steady state fractions in terms of μ'_D

$$\mu'_D = h - 1 = \frac{1}{\theta' + p'/\eta + 1}. \quad (28)$$

Finally, the measure of high valuation owners is given by $\mu'_H = p'\mu'_D/\eta$.

C Equilibrium existence conditions

Proof of Proposition 1.

1. Remember that sellers cannot exclude low valuation owners from showing up at their house, or that they cannot save on search costs by doing so. If the seller does not immediately sell her house at a discount to a low valuation owner that shows up, it is only worthwhile to continue looking for a renter. Otherwise, since the environment is static, the house could better be sold to the low valuation owner immediately. Denoting the continuation value of a double owner searching for a renter by V_D^R , double owners sell to both types of buyers if and only if

$$\begin{aligned} r(V_H + P_L) &\geq rV_D^R = v + u + p \frac{\mu_R}{\mu_B} (V_H - V_D^R + P_R) - c - F \\ \Leftrightarrow P_R + P_L &\geq V_D^R = \frac{v + u - c - F + 2p \frac{\mu_R}{\mu_B} P_R}{r + p \frac{\mu_R}{\mu_B}} \end{aligned} \quad (29)$$

$$\begin{aligned} \Leftrightarrow (r + p \frac{\mu_R}{\mu_B})(P_R + P_L) &\geq v + u - c - F + 2p \frac{\mu_R}{\mu_B} P_R \\ \Leftrightarrow r(P_R + P_L) &\geq v + u - c - F + p \frac{\mu_R}{\mu_B} (P_R - P_L) \\ \Leftrightarrow \frac{v - F + (r + 2p)P_R}{r + p} &\geq \frac{v + u - c - F}{r} - \frac{p}{r + p} \frac{\mu_R}{\mu_B} \frac{v - F - rP_R}{r} \\ \Leftrightarrow v - F + (r + 2p)P_R &\geq p \frac{v + u - c - F}{r} + v + u - c - F - \frac{p \mu_R}{r \mu_B} (v - F - rP_R) \\ \Leftrightarrow \left(r + p + p \frac{\mu_L}{\mu_B} \right) P_R &\geq p \frac{\mu_L}{\mu_B} \frac{v - F}{r} + (r + p) \frac{u - c}{r} \\ \Leftrightarrow P_R &\geq \frac{p \frac{\mu_L}{\mu_B} \frac{v - F}{r} + (r + p) \frac{u - c}{r}}{r + p + p \frac{\mu_L}{\mu_B}} \\ \Leftrightarrow P_R &\geq \frac{\frac{p}{r + p} \frac{\mu_L}{\mu_B} \frac{v - F}{r} + \frac{u - c}{r}}{1 + \frac{p}{r + p} \frac{\mu_L}{\mu_B}}. \end{aligned} \quad (30)$$

Comparing inequality (30) with expression (11) for P_R , the inequality is always satisfied.

2. It is *not* in the interest of a high valuation owner to search to sell her house if

$$\begin{aligned} p \left(V_R - V_H + \frac{\mu_L}{\mu_B} P_L + \frac{\mu_R}{\mu_B} P_R \right) &\leq c \\ \Leftrightarrow p \frac{\mu_L}{\mu_B} (P_L - P_R) &\leq c. \end{aligned}$$

Since $P_R > P_L$ by Lemma 2, high valuation owners do not enter the market as sellers.

3. For double owners the benefits of search weakly dominate the costs if

$$\begin{aligned} c &\leq p(V_H - V_D + P) = rV_D - v - u + c + F \\ \Leftrightarrow V_D &\geq \frac{v + u - F}{r}. \end{aligned} \tag{31}$$

Since $P_L = V_D - V_L$ and since V_L is at least u/r due to the max-functions, V_D exceeds u/r if $P_L > 0$. With Assumption 1, the participation constraint of double owners is therefore satisfied if P_L is positive.

Proof of Proposition 2.

1. It follows from the ranking of prices by Lemma 2, and the equality of expected prices $P_{i,L}$ and $P_{i,D}$ that

$$\begin{aligned} P_L &< P_R \\ \Leftrightarrow V_D - V_L &< V_H - V_R \\ \Leftrightarrow V_R - V_L &< V_H - V_D \\ \Leftrightarrow p(V_R - V_L + P_{i,L}) - c &< p(V_H - V_D + P_{i,D}) - c, \end{aligned} \tag{32}$$

which are the participation constraints of low valuation and double owners respectively. Consequently, the net benefits of search for double owners exceed those for low valuation owners.

2. Denoting the continuation value of a low valuation owner searching for a renter by V_L^R , low valuation owners sell to both types of buyers if and only if

$$\begin{aligned} r(V_R + P_L) &\geq rV_L^R = u + p \frac{\mu_R}{\mu_B} (V_R - V_L^R + P_R) - c \\ \Leftrightarrow P_L &\geq V_L^R = \frac{p \frac{\mu_R}{\mu_B} P_R + u - c}{r + p \frac{\mu_R}{\mu_B}}. \end{aligned} \quad (33)$$

To see that this condition is stronger than the one for double owners, subtract (33) from (29), to obtain

$$\begin{aligned} P_R &\geq \frac{v - F + p \frac{\mu_R}{\mu_B} P_R}{r + p \frac{\mu_R}{\mu_B}} \\ \Leftrightarrow \left(r + p \frac{\mu_R}{\mu_B} \right) P_R &\geq v - F + p \frac{\mu_R}{\mu_B} P_R \\ \Leftrightarrow r P_R &\geq v - F, \end{aligned}$$

which is always satisfied under Assumption 1.

Proof of Lemma 5.

1. Again I assume that a high valuation owner indifferent to buy a second house does not carry out this transaction in equilibrium. More importantly, it is not in her interest to search to sell her house, if low valuation owners do not enter the market as sellers either. This implication can be seen from

$$\begin{aligned} v/(r + \eta) &> 0 \\ \Leftrightarrow V_H' &> V_L' \\ \Leftrightarrow p'(V_R' - V_L' + P_L') - c &> p'(V_R' - V_H' + P_L') - c, \end{aligned}$$

so that the net benefits of search of low valuation owners always dominate those of high valuation owners.

2. Double owners search if the benefits are not smaller than the costs, thus if

$$p(V'_H - V'_D + P'_L) \geq c$$

$$\Leftrightarrow p'(V'_H - V'_L) \geq c$$

$$\Leftrightarrow p' \frac{v}{r + \eta} \geq c,$$

which, given Assumption 1, is a necessary condition for P'_L to be nonnegative.